Category Theory — Homework 4 Solutions — Name:

Dr. Paul L. Bailey — Sunday, September 29, 2019

Problem 1. A subgroup of $\mathbb{Z}_{91}^*$ contains $\{1, 9, 16, 22, 53, 74, 79, 81, x\}$. Find x.

Solution. Note that $22^2 = 484 = 29$ modulo 91. So, $x = 29$. □

Problem 2. Let G be a group such that $g^2 = 1$ for every $g \in G$. Show that G is abelian.

Solution. Let $g, h \in G$; then $g^2 = 1$ and $h^2 = 1$, whence $g^2h^2 = 1$. But $gh \in G$, so $(gh)^2 = 1$; that is, $ghgh = 1$. So $g^2h^2 = ghgh$. Cancelling g on the left and h on the right gives $gh = hg$. □

Problem 3. Let G be a finite group. Show that the number of elements $g \in G$ such that $g^3 = 1$ is odd.

Solution. Let $X = \{x \in G \mid x^3 = 1\}$. Partition X into blocks consisting of elements and their inverses. Since $1 \in X$, one of these blocks is $\{1\}$. Each other block contains two distinct elements. Thus $|X \setminus \{1\}|$ is even, so $|X|$ is odd. □

Problem 4. Let G be a group such that, for all $a, b, c, d, x \in G$, we have

$$axb = cxd \quad \Rightarrow \quad ab = cd.$$

Show that G is abelian.

Proof. Let $g, h \in G$. We wish to demonstrate that $gh = hg$.

Let $a = d = g$, $b = c = h$, and $x = g^{-1}$. Then

$$axb = gg^{-1}h = h = hg^{-1}g = cxd,$$

so by hypothesis, $ab = cd$; that is, $gh = hg$. □

Definition 1. Let G be a group and let $h \in G$. The *centralizer* of h in G is

$$C_G(h) = \{g \in G \mid gh = hg\}.$$

Problem 5. Let G a group and let $h \in G$. Show that $C_G(h)$ is a subgroup of G.

Solution. To prove that $C_G(h)$ is a subgroup, we show **(S0)**, **(S1)**, and **(S2)**.

(S0) Since $1 \cdot h = h \cdot 1$, we have $1 \in C_G(h)$.

(S1) Let $g_1, g_2 \in C_G(h)$. Then $g_1g_2h = g_1hg_2 = hg_1g_2$. Thus $g_1g_2 \in C_G(h)$.

(S2) Let $g \in C_G(h)$. Then $gh = hg$. Multiply by g^{-1} on the left and on the right to get $hg^{-1} = g^{-1}h$. Thus $g^{-1} \in C_G(h)$. □